

3 (Sem-1/CBCS) STA HC 2

2019

STATISTICS

(Honours)

Paper : STA-HC-1026

(Calculus)

Full Marks : 60

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. Choose the correct option for each of the following : 1×7=7

(a) According to L' Hospital's rule for indeterminate form $\frac{\infty}{\infty}$, under certain

conditions imposed upon the functions

$f(x)$ and $g(x)$, if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = l$, then the

value of $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is

(i) l

(ii) $\frac{1}{l}$

(iii) 0

(b) The gamma function $\int_0^{\infty} x^{m-1} e^{-x} dx$

converges for

(i) $m > 1$

(ii) $m > 0$

(iii) all real values of m

(c) The order and degree of the differential equation

$$\frac{x+y \frac{dy}{dx}}{x \frac{dy}{dx} - y} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$$

is

(i) 1, 1

(ii) 1, 2

(iii) 2, 1

(d) Extreme value of a function exists if and only if the first non-zero derivative of the function is of ____ order.

(i) odd

(ii) even

(e) The 'integrating factor' (in context of obtaining solution of a differential equation) of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

is

(i) $\log(\sec x)$ (ii) $\tan x$ (iii) $\sec x$

(f) Which of the following is not a linear partial differential equation?

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

(ii) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$

(iii) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

(g) The general solution of the linear differential equation

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

when the auxiliary equation has imaginary roots is of the form

(i) $y = ce^{\alpha x} \cos(\beta x + \epsilon)$

(ii) $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(iii) $y = (c_1 + c_2 x) e^{\alpha x}$

Here, the symbols have their usual meanings.

2. Answer the following questions :

$$2 \times 4 = 8$$

(a) If $x = r \cos \theta$ and $y = r \sin \theta$, then find the value of $\frac{\partial^2 \theta}{\partial x \partial y}$.

(b) Form the partial differential equation by eliminating the arbitrary constants a and b from

$$(x-a)^2 + (y-b)^2 + z^2 = \lambda^2$$

(c) Show that

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8} \sqrt{\pi}$$

(d) Solve :

$$\left(\frac{y^2 z}{x}\right)p + xzq = y^2$$

3. Answer any three from the following questions :

$$5 \times 3 = 15$$

(a) Find the point of maxima and the minima of the function $x^3 - 12x^2 + 45x$ in the interval $[0, 7]$.

(b) Solve :

$$(x \cos x) \frac{dy}{dx} + y(x \sin x + \cos x) = 1$$

(c) Use the relation between gamma and beta function to show that

$$\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx = \frac{3\pi}{128}$$

(d) Evaluate the limit :

$$\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$$

(e) Solve the differential equation

$$\left(\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4\right) y = x^2$$

when $x = 0$, $y = \frac{3}{8}$ and $\frac{dy}{dx} \cdot y = 1$.

Answer the following questions :

$$10 \times 3 = 30$$

4. (a) (i) Evaluate :

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$$\frac{1}{\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2} e^{e^x}$$

(ii) Solve :

4

$$\frac{d^2 y}{dx^2} = \cos nx$$

Or

- (b) Define 'Jacobian'. If $u = \frac{x^2 + y^2 + z^2}{x}$,
 $v = \frac{x^2 + y^2 + z^2}{y}$ and $w = \frac{x^2 + y^2 + z^2}{z}$,
 then find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$. 3+7=10

5. (a) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1 \text{ and } z = x + y \quad 10$$

Or

- (b) (i) Solve the Clairaut's equation : 5

$$y = px + p - p^2, \text{ where } p = \frac{dy}{dx}$$

- (ii) Show that the differential equation

$$xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$$

is exact and hence solve it. 5

6. (a) (i) Evaluate the integral : 5

$$\iint_R (x^2 + 2y) dx dy, R = [0, 1; 0, 2]$$

- (ii) Prove that a necessary and sufficient condition that the differential equation $Mdx + Ndy = 0$ be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad 5$$

Or

- (b) (i) Find the first- and second-order partial derivatives of $z = x^3 + y^3 - 3axy$ and verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad 2+3=5$$

- (ii) Given $\int_0^\infty \frac{xp^{-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, show that

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \text{ when } 0 < p < 1 \quad 5$$

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Total number of printed pages-8

3 (Sem - 1 /CBCS) STA HC 2

2021

(Held in 2022)

STATISTICS

(Honours)

Paper : STA-HC- 1026

(Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed :

1×10=10

(a) Define differential coefficient of $f(x)$ at the point $x=a$.

(b) The value of $\lim_{x \rightarrow 0} \frac{\tan x}{x}$ is

(i) 0

(ii) 1

(iii) α

(iv) None of the above

(Choose the correct option)

Contd.

(c) Evaluate $\Gamma\left(-\frac{3}{2}\right)$.

(d) State Leibnitz's theorem.

(e) Show that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

(f) Find the differential equation of lines parallel to x-axis.

(g) The integral $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

converges if

(i) $m > 0, n > 0$

(ii) $m < 0, n > 0$

(iii) $m > -1, n > -1$

(Choose the correct option)

(h) If $f(x, y) = 2x^2 - xy + 2y^2$, then find

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (1, 2).

(i) The differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 - 2\left(\frac{dy}{dx}\right)^2 + 5y = 0 \text{ is}$$

(i) an ordinary differential equation

(ii) of order two and degree two

(iii) called partial differential equation
(Choose the incorrect option)

(j) Find the value of

$$\lim_{x \rightarrow a} \frac{x^4}{e^x}$$

2. Answer the following questions : $2 \times 5 = 10$

(a) Examine the differentiability at $x=0$ of the function f defined on the set of real number as follows :

$$f(x) = x^2 \sin \frac{1}{x}, \text{ if } x \neq 0 \\ = 0, \text{ if } x = 0$$

(b) Evaluate $\lim_{x \rightarrow 0} (\sin x \log x)$

(c) Show that $f(x) = x^3 - 6x^2 + 24x + 1$ has neither a maximum nor a minimum.

- (d) Obtain a differential equation from the relation

$$y = A \sin x + B \cos x + x \sin x$$

- (e) Show that for $l > 0, m > 0$

$$\int_a^b (x-a)^{l-1} (b-x)^{m-1} dx = (b-a)^{l+m-1} \beta(l, m)$$

3. Answer **any four** from the following questions :

$$5 \times 4 = 20$$

- (a) Show that if a function is differentiable at a point, then it is continuous at that point but the converse is not necessarily true.

- (b) Show that the necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- (c) Evaluate $\int_1^{\log 8} \int_1^{\log y} e^{x+y} dy dx$

- (d) If (a, b) be a point of the domain of definition of a function f such that

- (i) f_x is continuous at (a, b)

- (ii) f_y exists at (a, b) , then show f is differentiable at (a, b) .

- (e) If $u = \sin^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$, then using Euler's theorem show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u$$

- (f) Prove that $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

4. (a) (i) If $y = \sin^{-1} x$, then using Leibnitz's theorem prove that

$$(1-x^2) y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$$

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- (ii) Test the continuity and differentiability of the function

$$f(x) = \begin{cases} 1+x & \text{if } x \leq 2 \\ 5-x & \text{if } x \geq 2 \end{cases} \quad 4$$

at $x=2$

Or

- (b) Solve the differential equation

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \quad 10$$

5. (a) (i) For a positive number P , show that

$$\Gamma(P)\Gamma\left(P+\frac{1}{2}\right)2^{2P-1} = \sqrt{\pi} \Gamma(2P) \quad 6$$

(ii) Evaluate $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$ 4

Or

(b) (i) Evaluate $\int_0^{\pi/2} \log \sin x \, dx$ 5

- (ii) If $u = 2(ax+by)^2 - (x^2+y^2)$ and $a^2+b^2=1$, find the value of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad 5$$

6. (a) (i) Show that the function $u = x^3 + y^3 - 3axy$ has a maximum or minimum at the point (a,a) according as a is negative or positive. 5

(ii) If $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$;

$(x,y) \neq (0,0)$, $f(0,0) = 0$, then show that at the origin $f_{xy} \neq f_{yx}$. 5

Or

- (b) (i) Solve the differential equation :

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 5y = \sin x \quad 5$$

- (ii) Define Clairaut's equation. Explain the general solution of Clairaut's equation. 5

7. (a) (i) If $u^3 + v^3 = x + y$,
 $u^2 + v^2 = x^3 + y^3$, prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)} \quad 5$$

- (ii) Solve the partial differential equation : 5

$$\left(\frac{y^2 z}{x} \right) P + xzq = y^2$$

Or

- (b) If f is defined and continuous on the rectangle $R = [a, b; c, d]$, and if

- (i) $f_x(x, y)$ exists and is continuous on the rectangle R , and

- (ii) $g(x) = \int_c^d f(x, y) dy$ for $x \in [a, b]$
 then show that g is differentiable

on $[a, b]$ and $g'(x) = \int_c^d f_x(x, y) dy$

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